Multiperiod Competitive Supply Chain Networks with Inventorying and A Transportation Network Equilibrium Reformulation

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Abstract: In this paper, we present a multitiered dynamic supply chain network equilibrium modeling framework in which the decision-makers have sufficient information about the future and seek to determine their optimal plans that maximize their profits over the multiperiod planning horizon. We construct the finite-dimensional variational inequality governing the equilibrium of the multiperiod competitive supply chain network. The model allows us to investigate the interplay of the heterogeneous decision-makers in the supply chain in a dynamic setting, and to compute the resultant equilibrium pattern of product outputs, transactions, inventories, and product prices. We then establish the supernetwork equivalence of the multiperiod supply chain model with a properly configured transportation network, which provides a new interpretation of the equilibrium conditions of the former in terms of paths and path flows. This framework offers great modeling flexibility so that, for example, transportation delay and/or perishable products can be easily handled, as we also demonstrate. Numerical examples are provided to illustrate how such multiperiod supply chain problems can be reformulated and solved as transportation network equilibrium problems in practice.

Key words: supply chains, transportation, network equilibrium, pricing, variational inequalities, multiperiod decision-making

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1. Introduction

A supply chain is a network of manufacturers, storage facility managers, transporters and retailers that perform the functions of production, storage, transportation, and sale of a particular product. Nowadays, the highly dynamic and competitive business environment makes the decision-making of the supply chain participants increasingly complex. In order to survive and thrive, the business enterprises need to make intelligent and consistent decisions that not only provide optimal decisions today but also benefit them in the future. In this paper, we propose a multitiered, multiperiod supply chain network model which can be utilized to investigate and facilitate the complex decision-making of the supply chain participants in a competitive and dynamic global environment.

The majority of the supply chain management literature has focused primarily on the optimization problems faced by a single decision-maker in the supply chain (cf. Federgruen and Zipkin (1986), Federgruen (1993), Lee and Billington (1993), Slats et al. (1995), Anupindi and Bassok (1996), Bramel and Simchi-Levi (1997), Ganeshan et al. (1998), Stadtler and Kilger (2000), Miller (2001), Mentzer (2001), Hensher et al. (2001)). More recently, decentralized decision-making and competition in the supply chain have been studied using game theory. Lederer and Li (1997) modeled the competition between the firms that provide products or services to the customers who are sensitive to the delay-time. Cachon and Zipkin (1999) studied inventorying decision-making in a two stage serial supply chain. Corbett and Karmarkar (2001) investigated a supply chain network consisting of several tiers of decision-makers and provided a framework for comparing a variety of supply chain structures. Bernstein and Federgruen (2003), in turn, modeled retail market competition in the case of a single supplier and multiple retailers in a multiperiod setting. The book chapter by Cachon and Netessine (2003) overviews applications of game theory to supply chain modeling and analysis. See also the annotated bibliography on network optimization in supply chains and financial engineering by Geunes and Pardalos (2003). More recently, Perakis and Sood (2006) studied multiperiod pricing at retail markets using a robust optimization approach.

Nagurney et al. (2002) proposed the first supply chain network equilibrium model, which was multitiered and involved competition among decision-makers in a given tier, but cooperation between tiers of decision-makers, consisting of manufacturers, retailers, and consumers at the demand markets. The governing equilibrium conditions were formulated as a finite-dimensional variational inequality problem. Recently, it was shown by Nagurney (2006a) that this problem can be reformulated and solved as a transportation network equilibrium problem in paths and path flows, which has opened up the study of supply chains through the prism of transportation networks, a subject with a much longer history and literature. Nagurney et al. (2005), subsequently, investigated the impact of the supply side and demand side risk on multitiered supply chain networks. The book by Nagurney (2006b) describes the numerous applications of multitiered network problems with a supply chain foundation, ranging from a variety of static and dynamic supply chains to electric power generation and distribution networks as well as to financial networks with intermediation, with a focus on their relationships to transportation network equilibrium problems.

Beckmann et al. (1956) proposed the first rigorous mathematical treatment of transportation network equilibrium problems in their classic book, Studies in the Economics of Transportation. For additional research highlights in transportation network equilibrium, see Boyce et al. (2005), Florian and Hearn (1995), and the books by Patriksson (1994) and Nagurney (1999, 2000).

Interestingly, earlier to the identification of supply chain network problems with transportation network problems, Dafermos and Nagurney (1985) and Dafermos (1986) demonstrated (see also Dafermos and Nagurney (1984a)) that spatial price equilibrium problems (cf. Samuelson (1952) and Takayama and Judge (1971)) could be transformed into transportation network equilibrium problems over appropriately constructed abstract networks, now commonly referred to as *supernetworks* (see also, e.g., Nagurney and Dong (2002)). Subsequently, Nagurney and Aronson (1988) developed a multiperiod spatial pricing equilibrium model where inventorying and backordering were allowed at both the supply markets and the demand markets. Nagurney and Aronson (1989) then extended that research and presented a general dynamic spatial price equilibrium model with gains and losses which was capable of handling directly agriculture markets with perishable commodities as well as financial markets.

In this paper, we develop a multiperiod competitive supply chain network equilibrium model in which the manufacturers, the retailers, and the consumers associated with the demand markets are located at distinct tiers of the network, and decisions are made in discrete time periods over a finite planning horizon. The manufacturers produce a homogenous product and sell to the retailers. We assume that each manufacturer has sufficient information about the future, and seeks the optimal production, transaction, and inventory plan in order to maximize his total profit over the planning horizon. We also assume that the manufacturers compete in a noncooperative manner in the sense of Nash (1950, 1951).

The retailers, in turn, purchase the products from the manufacturers and sell to the consumers at the demand markets. Each retailer seeks the optimal replenishment and inventory plan to maximize his profit over the planning horizon. We assume that the retailers also have sufficient information about the future and compete with other retailers in a noncooperative manner.

Finally, at each time period, the consumers at the various demand markets determine their consumption levels, and take into consideration both the prices charged by the retailers and the unit transaction/transportation costs in making their consumption decisions. We allow the specifications of the demand functions to change in different time periods so that different trends or seasonalities of the demand can be captured.

The equilibrium state of the multiperiod competitive supply chain network is one where the manufacturers and the retailers achieve optimality over the entire planning horizon, and the equilibrium conditions at the demand markets are satisfied at each period so that no decision-maker has any incentive to alter his decisions. We formulate the governing equilibrium conditions as a finite-dimensional variational inequality, and prove that this dynamic supply chain network equilibrium model is isomorphic to a properly configured transportation network equilibrium model with elastic demand (cf. Dafermos (1982) and Dafermos and Nagurney (1984b)). This mathematical equivalence provides a new economic interpretation for the multiperiod supply chain network equilibrium in terms of paths and path flows, and also allows us to transfer the methodological tools developed for transportation network equilibrium modeling, analysis, and computation to the study of such supply chains. It is worth noting that since equilibria for large-scale transportation networks are routinely computed in practice, the results herein also suggest new opportunities for the effective solution of large-scale multiperiod supply chain networks with inventorying.

This paper is organized as follows. In Section 2, we present the multitiered, multiperiod supply chain network model, and establish the finite-dimensional variational inequality governing the equilibrium. In Section 3, we recall the transportation network equilibrium model with elastic demands of Dafermos and Nagurney (1984b), which was also studied by Nagurney and Zhang (1996). In Section 4, we establish that the proposed multiperiod supply chain network equilibrium model of Section 2 can be reformulated as a transportation network equilibrium model as described in Section 3, over a properly constructed abstract network or *supernetwork* (cf. Nagurney and Dong (2002) and the references therein). We also discuss how this model can capture time delays associated with transportation as well as perishable products. In Section 5, numerical examples of multiperiod supply chain networks are reformulated and solved as transportation networks using algorithms developed for the computation of transportation network equilibria. In Section 6, we present a summary of the results of this paper, along with our conclusions.

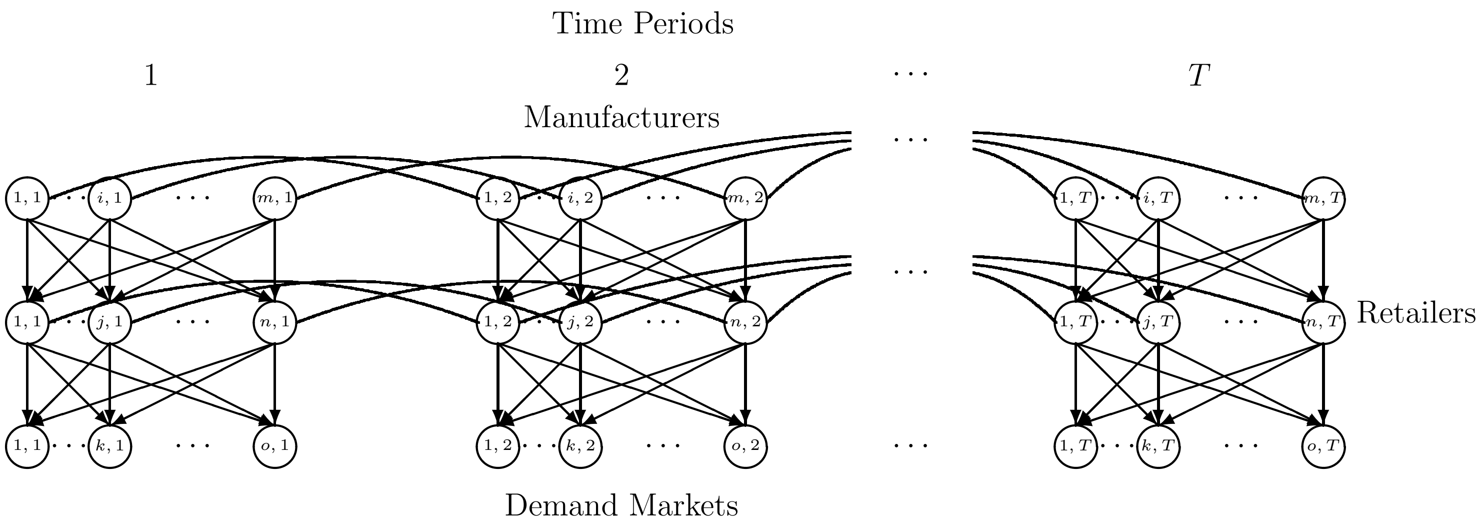

Figure 1: The Network Structure of the Multiperiod Supply Chain

2. The Multiperiod Supply Chain Network Model

In this section, we develop the multiperiod supply chain network model with elastic demands. The time planning horizon is discretized into periods: $1, ..., t, ..., T$. The model consists of m manufacturers, n retailers, and o demand markets, as depicted in Figure 1. We denote a typical manufacturer by i, a typical retailer by j, and a typical demand market by k. In the supply chain network, the links between the tiers represent transportation/transaction links while the links between the adjacent temporal subnetworks represent the inventory links. The majority of the needed notation is shown in Tables 1 and 2. The equilibrium solution is denoted by "*". All vectors are assumed to be column vectors, except where noted.

The top-tiered nodes in Figure 1 represent the m manufacturers in the T time periods with node (i, t) denoting manufacturer i in time period t. The manufacturers are the decision-makers who produce a homogeneous product and sell to the retailers in the second tier of nodes in the supply chain network in Figure 1. A node (j, t) corresponds to retailer j in time period t, where $j = 1, \ldots, n$ and $t = 1, \ldots, T$. Both the manufacturers and the retailers can store the product in their inventories with a link joining node (i, t) with node $(i, t + 1)$ in

Table 1: Variables in the Multiperiod Supply Chain Network Equilibrium Model

Notation	Definition
q	mT-dimensional vector of the manufacturers' production outputs during the entire planning horizon with component it: q_{it}
q_t	m-dimensional vector of the manufacturers' production outputs at time period t with component i: q_{it}
Q^1	mnT-dimensional vector of product flows transacted/shipped between manufacturers and retailers with component ijt: q_{ijt}
Q^1_t	mn-dimensional vector of product flows transacted/shipped between manufacturers and retailers at time period t with component ij: q_{ijt}
Q^2	noT-dimensional vector of product flows transacted/shipped between retailers and the demand markets during the entire planning horizon with component jkt: q_{jkt}
Q^2_t	no-dimensional vector of product flows transacted/shipped between retailers and the demand markets at time period t with component jk: q_{jkt}
h	nT-dimensional vector of the retailers' supplies of the product during the planning horizon with component jt: h_{jt}
h_t	n-dimensional vector of the retailers' supplies of the product at time period t with component j: h_{jt}
u^1	mT-dimensional vector of the manufacturers' inventory levels during the planning horizon with component it: u_{it}
u^2	nT-dimensional vector of the retailers' inventory levels during the planning horizon with component jt: u_{jt}
d	oT-dimensional vector of satisfied market demand with component kt: d_{kt}
ρ_1	mnT-dimensional vector of prices charged by the manufacturers in transacting with the retailers during the entire planning horizon with component ijt: ρ_{1ijt}
ρ_2	nT-dimensional vector of prices charged by the retailers during the entire planning horizon with component jt: ρ_{2jt}
ρ_3	kT-dimensional vector of prices of the product at the demand markets with component kt: ρ_{3kt}
ρ_{3t}	k-dimensional vector of prices of the product at the demand markets at time period t with component k: ρ_{3kt}

Table 2: Demand and Cost Functions in the Multiperiod Supply Chain Network Equilibrium Model

Notation	**Definition**
$d_{kt}(\rho_{3t})$	demand function at demand market k at time period t
$f_{it}(q)$	production cost of manufacturer i at period t with marginal production cost with respect to q_{it}: $\frac{\partial f_{it}}{\partial q_{it}}$
$c_{ijt}(q_{ijt})$	transaction cost between manufacturer i and retailer j at period t with marginal transaction cost: $\frac{\partial c_{ijt}(q_{ijt})}{\partial q_{ijt}}$
$c_{jt}(h_t)$	handling cost of retailer j at time period t with marginal handling cost with respect to h_{jt}: $\frac{\partial c_{jt}}{\partial h_{jt}}$
$cv_{it}(u_{it})$	inventory cost of manufacturer i at time period t with marginal inventory cost with respect to u_{it}: $\frac{\partial cv_{it}}{\partial u_{it}}$
$cv_{jt}(u_{jt})$	inventory cost of retailer j at time period t with marginal inventory cost with respect to u_{jt}: $\frac{\partial cv_{jt}}{\partial u_{jt}}$
$\hat{c}_{jkt}(Q_t^2)$	unit transaction cost between retailer j and demand market k at time period t

Figure 1 corresponding to the inventorying of manufacturer i between time period t and $t+1$ and the link joining node (j,t) with node $(j,t+1)$ corresponding to the inventorying of the product by retailer j from time period t to time period $t+1$. The consumers at the demand markets are represented by the nodes in the bottom tier of the supply chain network and they acquire the product from the retailers. A demand market k at time period t is denoted by node (k,t) with $k=1,\ldots,o$ and $t=1,\ldots,T$.

We first describe the behavior of the manufacturers and then that of the retailers. We, subsequently, discuss the behavior of the consumers at the demand markets. Finally, we state the equilibrium conditions for the multiperiod supply chain network and provide the finite-dimensional variational inequality governing the equilibrium.

2.1 The Behavior of the Manufacturers and their Optimality Conditions

Let ρ_{1ijt}^* denote the price charged for the product by manufacturer i in transacting with retailer j in period t. The price ρ_{1ijt}^* is an endogenous variable and will be determined once the entire multiperiod supply chain network equilibrium model is solved. We assume that the quantity produced by manufacturer i must satisfy the following conservation of flow equations:

$$\sum_{j=1}^{n} q_{ij1} + u_{i1} = q_{i1}, \tag{1}$$

$$u_{it} + \sum_{j=1}^{n} q_{ijt} = q_{it} + u_{i(t-1)}, \quad t = 2, \ldots, T-1, \tag{2}$$

$$\sum_{j=1}^{n} q_{ijT} = q_{iT} + u_{i(T-1)}. \tag{3}$$

Constraints (1), (2), and (3) state that at each period, the amount of product available for distribution at that time period and inventorying to the next period, is equal to the amount produced in that period plus the amount inventoried from the preceding period, with zero inventories assumed before the first time period and after the final time period T.

The objective of the manufacturers is to maximize the total profit over the planning horizon T. The decision variables for manufacturer i are: the production levels at each period, q_t; $t = 1, ..., T$, the distribution quantities in each period, q_{ijt}; $j = 1, ..., n; t = 1, ..., T$, and the inventory level at the end of each period, u_{it}; $t = 1, ..., T-1$. Thus, manufacturer i is faced with an optimization problem which can be expressed as follows:

$$\text{Maximize} \quad \sum_{t=1}^{T}\sum_{j=1}^{n} \rho^*_{1ijt} q_{ijt} - \sum_{t=1}^{T} f_{it}(q_t) - \sum_{t=1}^{T} cv_{it}(u_{it}) - \sum_{t=1}^{T}\sum_{j=1}^{n} c_{ijt}(q_{ijt}) \tag{4}$$

subject to (1), (2), and (3), and

$$q_{ijt} \geq 0, \quad j = 1, \ldots, n; \quad t = 1, \ldots, T,$$

$$q_{it} \geq 0, \quad t = 1, \ldots, T,$$

$$u_{it} \geq 0, \quad t = 1, \ldots, T-1.$$

The first term in (4) represents the revenue and the subsequent three terms the production costs, inventory costs, and the transaction costs, respectively, for manufacturer i. Note that we allow the specifications of all the cost functions to be time-dependent.

We assume that the production cost functions f_{it}; $i = 1, \ldots, m$; $t = 1, \ldots, T$ are continuously differentiable and convex as are the transaction cost functions, c_{ijt}; $i = 1, \ldots, m$; $j = 1, \ldots, n$; $t = 1, \ldots, T$, and the inventory cost functions cv_{it}; $i = 1, \ldots, m$; $t = 1, \ldots, T$.

We assume that the manufacturers compete in a noncooperative manner in the sense of Nash (1950, 1951) (see also, e.g., Nagurney et al. (2002) and Nagurney et al. (2005)). The optimality conditions for all manufacturers i; $i = 1, \ldots, m$, simultaneously, can then be expressed as the following variational inequality (cf. Nagurney et al. (2002), Bazaraa et al. (1993), Gabay and Moulin (1980); see also Dafermos and Nagurney (1987) and Nagurney (1999)): determine $(q^*, u^{1*}, Q^{1*}) \in \mathcal{K}^1$ satisfying:

$$\sum_{t=1}^{T}\sum_{i=1}^{m} \frac{\partial f_{it}(q_t^*)}{\partial q_{it}} \times [q_{it} - q_{it}^*] + \sum_{t=1}^{T}\sum_{i=1}^{m}\sum_{j=1}^{n} \left[\frac{\partial c_{ijt}(q_{ijt}^*)}{\partial q_{ijt}} - \rho_{1ijt}^*\right] \times \left[q_{ijt} - q_{ijt}^*\right]$$

$$+\sum_{t=1}^{T}\sum_{i=1}^{m} \frac{\partial cv_{it}(u_{it}^*)}{\partial u_{it}} \times [u_{it} - u_{it}^*] \geq 0, \quad \forall (q, u^1, Q^1) \in \mathcal{K}^1, \tag{5}$$

where $\mathcal{K}^1 \equiv \{(q, u^1, Q^1) | (q, u^1, Q^1) \in R_+^{Tm(2+n)} \text{ and } (1),\ (2),\ \text{and } (3) \text{ hold}\}$.

2.2 The Behavior of the Retailers and their Optimality Conditions

The retailers, in turn, purchase the product from the manufacturers and transact with the consumers at the demand markets. Thus, a retailer is involved in transactions both with the manufacturers as well as with customers at the demand markets.

Let ρ_{2jt}^* denote the price charged by retailer j for the product at time period t. This price will be determined endogenously after the complete model is solved. We assume that the objective of a retailer is to maximize his total profit over the planning horizon T. The decision variables of retailer j include: the procurement in each period, q_{ijt}; $i = 1, ..., m; t = 1, ..., T$, the sales made at each period, q_{jkt}; $k = 1, ..., o; t = 1, ..., T$, and the inventory level at the end of each period, u_{jt}; $t = 1, ..., T-1$. Hence, the optimization problem faced by retailer j is given by:

$$\text{Maximize} \quad \sum_{t=1}^{T}\sum_{k=1}^{o} \rho_{2jt}^* q_{jkt} - \sum_{t=1}^{T} c_{jt}(h_t) - \sum_{t=1}^{T} cv_{jt}(u_{jt}) - \sum_{t=1}^{T}\sum_{i=1}^{m} \rho_{1ijt}^* q_{ijt} \tag{6}$$

subject to:

$$h_{jt} = \sum_{i=1}^{m} q_{ijt}, \quad t = 1, ..., T, \tag{7}$$

$$\sum_{k=1}^{o} q_{jk1} + u_{j1} = \sum_{i=1}^{m} q_{ij1}, \tag{8}$$

$$\sum_{k=1}^{o} q_{jkt} + u_{jt} = \sum_{i=1}^{m} q_{ijt} + u_{j(t-1)}, \quad t = 2, ..., T-1, \tag{9}$$

$$\sum_{k=1}^{o} q_{jkT} = \sum_{i=1}^{m} q_{ijT} + u_{j(T-1)}, \tag{10}$$

and the nonnegativity constraints:

$$q_{ijt} \geq 0, \quad i = 1, \ldots, m; t = 1, \ldots, T,$$

$$q_{jkt} \geq 0, \quad k = 1, \ldots, m; t = 1, \ldots, T,$$

$$u_{jt} \geq 0, \quad t = 1, \ldots, T.$$

The first term in the objective function (6) represents the revenue of retailer j, whereas the second, third, and the fourth terms represent, respectively, the handling cost, the inventory cost and the payout to the manufacturers. Constraints (8), (9), and (10) state that the amount available of the product for distribution to the demand markets in a time period is equal to the amount obtained in that period from the manufacturers plus the amount inventoried from the preceding period minus the amount inventoried for the next time period. Constraint (7) is a notational constraint that will be useful in our transportation network equivalence.

We assume that the handling cost functions for each retailer c_{jt}; $j = 1, \ldots, n$; $t = 1, ...T$, are continuously differentiable and convex as are the inventory cost functions cv_{jt}; $j = 1, \ldots, n$; $t = 1, \ldots, T$.

We assume that the retailers also compete in a noncooperative manner. Then the optimality conditions for all the retailers simultaneously can be expressed as the variational inequality: determine $(Q^{1*}, h^*, u^{2*}, Q^{2*}) \in \mathcal{K}^2$ satisfying:

$$\sum_{t=1}^{T}\sum_{j=1}^{n} \frac{\partial c_{jt}(h_t^*)}{\partial h_{jt}} \times \left[h_{jt} - h_{jt}^*\right] + \sum_{t=1}^{T}\sum_{i=1}^{m}\sum_{j=1}^{n} \rho_{1ijt}^* \times \left[q_{ijt} - q_{ijt}^*\right] - \sum_{t=1}^{T}\sum_{j=1}^{n}\sum_{k=1}^{o} \rho_{2jt}^* \times \left[q_{jkt} - q_{jkt}^*\right]$$

$$+\sum_{t=1}^{T}\sum_{j=1}^{n}\frac{\partial cv_{jt}(u^*_{jt})}{\partial u_{jt}}\times\left[u_{jt}-u^*_{jt}\right]\geq 0,\quad \forall (Q^1,h,u^2,Q^2)\in\mathcal{K}^2, \tag{11}$$

where $\mathcal{K}^2\equiv\{(Q^1,h,u^2,Q^1)|(Q^1,h,u^2,Q^1)\in R_+^{Tn(m+o+2)}$ and $(7),\ (8),\ (9),\ \text{and}\ (10)$ hold$\}$.

The Consumers at the Demand Markets and the Equilibrium Conditions

We now describe the behavior of the consumers located at the demand markets. The consumers take into account in making their consumption decisions not only the prices charged for the product by the retailers, ρ^*_{2jt}; $j=1,\ldots,n$; $t=1,\ldots,T$, but also on the unit transaction costs to obtain the product. The equilibrium conditions for consumers at demand market k, (cf. Samuelson (1952) and Takayama and Judge (1971)) take the form: for all retailers j; $j=1,\ldots,n$ and time periods t; $t=1,\ldots,T$:

$$\rho^*_{2jt}+\hat{c}_{jkt}(Q^{2^*}_t)\begin{cases}=\rho^*_{3kt}, & \text{if}\quad q^*_{jkt}>0,\\ \geq\rho^*_{3kt}, & \text{if}\quad q^*_{jkt}=0,\end{cases} \tag{12}$$

and

$$d_{kt}(\rho^*_{3t})\begin{cases}=\sum_{j=1}^{n}q^*_{jkt}, & \text{if}\quad \rho^*_{3kt}>0,\\ \leq\sum_{j=1}^{n}q^*_{jkt}, & \text{if}\quad \rho^*_{3kt}=0.\end{cases} \tag{13}$$

Note that we allow the specification of the elastic demand function $d_{kt}(\rho^*_{3t})$ to be time-dependent.

Conditions (12) state that, in equilibrium, at each time period, if the consumers at demand market k purchase the product from retailer j, then the price charged by the retailer for the product at that time period plus the unit transaction cost is equal to the price that the consumers are willing to pay for the product at that time period. If the price plus the unit transaction cost is higher than the price the consumers are willing to pay at the demand market then there will be no transaction between the retailer and demand market pair at that time period. Conditions (13) state, in turn, that if the equilibrium price the consumers are willing to pay for the product at the demand market at the time period is positive, then the quantities purchased of the product from the retailers at that time period will be precisely equal to the demand for that product at the demand market at that time period.

If the equilibrium price at the demand market is zero at the time period then the shipments to that demand market may exceed the actually demand at the time period.

For notational convenience (see also Table 1), we let:

$$d_{kt} = \sum_{j=1}^{n} q_{jkt} \quad k = 1, ..., o; \quad t = 1, ..., T. \tag{14}$$

In equilibrium, condition (12) and (13) must hold simultaneously for all demand markets k; $k = 1, \dots, o$ at all the time periods. We can also express these equilibrium conditions using the following variational inequality: determine $(Q^{2*}, d^*, \rho_3^*) \in \mathcal{K}^3$, such that

$$\sum_{t=1}^{T}\sum_{j=1}^{n}\sum_{k=1}^{o} \hat{c}_{jkt}(Q_t^{2*}) \times \left[q_{jkt} - q_{jkt}^*\right] + \sum_{t=1}^{T}\sum_{k=1}^{o} \rho_{3kt}^* \times [d_{kt} - d_{kt}^*]$$

$$+ \sum_{t=1}^{T}\sum_{k=1}^{o} [d_{kt}^* - d_{kt}(\rho_{3t}^*)] \times [\rho_{3kt} - \rho_{3kt}^*] \geq 0, \quad \forall (Q^2, d, \rho_3) \in \mathcal{K}^3, \tag{15}$$

where $\mathcal{K}^3 \equiv \{(Q^2, d, \rho_3) | (Q^2, d, \rho_3) \in R_+^{T(no+2o)}$ and (14) holds$\}$.

The Equilibrium Conditions of the Multiperiod Supply Chain Network

In equilibrium, the optimality conditions for all manufacturers, the optimality conditions for all retailers, and the equilibrium conditions for all the demand markets must hold simultaneously so that no decision-maker can be better off by altering his decisions. Also, the shipments that the manufacturers ship to the retailers must be equal to the shipments that the retailers accept from the manufacturers. Similarly, the quantities of the product obtained by the consumers at the demand markets must coincide with the amounts sold by the retailers.

Definition 1: Multiperiod Supply Chain Network Equilibrium

The equilibrium state of the multiperiod supply chain network is one where the sum of (5), (11), and (15) is satisfied, so that no decision-maker has any incentive to alter his decisions.

We now state Theorem 1.

Theorem 1: Variational Inequality Formulation

The equilibrium conditions governing the multiperiod supply chain network model are equivalent to the solution of the variational inequality problem given by: determine

$$(q^*, h^*, u^{1*}, Q^{1^*}, u^{2*}, Q^{2^*}, d^*, \rho_3^*) \in \mathcal{K}^4$$

satisfying:

$$\sum_{t=1}^{T}\sum_{i=1}^{m}\frac{\partial f_{it}(q_t^*)}{\partial q_{it}} \times [q_{it} - q_{it}^*] + \sum_{t=1}^{T}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\partial c_{ijt}(q_{ijt}^*)}{\partial q_{ijt}} \times \left[q_{ijt} - q_{ijt}^*\right] + \sum_{t=1}^{T}\sum_{j=1}^{n}\frac{\partial c_{jt}(h_t^*)}{\partial h_{jt}} \times \left[h_{jt} - h_{jt}^*\right]$$

$$+\sum_{t=1}^{T}\sum_{i=1}^{m}\frac{\partial cv_{it}(u_{it}^*)}{\partial u_{it}} \times [u_{it} - u_{it}^*] + \sum_{t=1}^{T}\sum_{j=1}^{n}\frac{\partial cv_{jt}(u_{jt}^*)}{\partial u_{jt}} \times \left[u_{jt} - u_{jt}^*\right]$$

$$+\sum_{t=1}^{T}\sum_{j=1}^{n}\sum_{k=1}^{o}\hat{c}_{jkt}(Q_t^{2*}) \times \left[q_{jkt} - q_{jkt}^*\right] + \sum_{t=1}^{T}\sum_{k=1}^{o}\rho_{3kt}^* \times [d_{kt} - d_{kt}^*]$$

$$+\sum_{t=1}^{T}\sum_{k=1}^{o}[d_{kt}^* - d_{kt}(\rho_{3t}^*)] \times [\rho_{3kt} - \rho_{3kt}^*] \geq 0, \quad \forall (q, h, u^1, Q^1, u^2, Q^2, d, \rho_3) \in \mathcal{K}^4, \tag{16}$$

where $\mathcal{K}^4 \equiv \{(q, h, u^1, Q^1, u^2, Q^2, d, \rho_3)|$

$(q, h, u^1, Q^1, u^2, Q^2, d, \rho_3) \in R_+^{T(2m+mn+2n+no+2o)}$ and (1), (2), (3), (7), (8), (9), (10), and (14) hold$\}$.

Proof: See the Appendix.

3. The Transportation Network Equilibrium Model with Elastic Demands

In this section, we recall a transportation network equilibrium model with elastic demands. We assume that the demand functions associated with the origin/destination (O/D) pairs are given, and we provide the single-modal version of the model of Dafermos and Nagurney (1984b).

Consider a network G with the set of links L consisting of K elements, the set of paths P consisting of n_P elements. Let W denote the set of O/D pairs with n_W elements. Let P_w denote the set of paths connecting O/D pair w. Links are denoted by a, b, etc; paths by p, q, etc., and O/D pairs by w, ω, etc.

We denote the flow on path p by x_p and the flow on link a by f_a. We group the path flows into the vector x and the link flows into the vector f. We also denote the user travel cost on path p by C_p and the user travel cost on link a by c_a. The travel demand associated with traveling between O/D pair w is denoted by d_w and the travel disutility by λ_w.

We assume that the following conservation of flow equations hold:

$$f_a = \sum_{p \in P} x_p \delta_{ap}, \quad \forall a \in L, \tag{17}$$

where $\delta_{ap} = 1$ if link a is contained in path p, and $\delta_{ap} = 0$, otherwise. Expression (17) means that the flow on a link is equal to the sum of the flows on paths that contain that link.

The user travel cost on a path is equal to the sum of user travel costs on links that comprise the path:

$$C_p = \sum_{a \in L} c_a \delta_{ap}, \quad \forall p \in P. \tag{18}$$

Here we consider the general situation where the cost on a link may depend upon the entire vector of link flows, so that

$$c_a = c_a(f), \quad \forall a \in L. \tag{19}$$

We assume that the travel demand functions are given as follows:

$$d_w = d_w(\lambda), \quad \forall w \in W, \tag{20}$$

where λ is the vector of travel disutilities with the travel disutility associated with O/D pair being denoted by λ_w.

As given in Dafermos and Nagurney (1984b); see also Aashtiani and Magnanti (1981), Fisk and Boyce (1982), Nagurney and Zhang (1996), and Nagurney (1999), a travel path flow and disutility pattern $(x^*, \lambda^*) \in R_+^{n_P+n_W}$ is said to be an equilibrium, if, once established, no user can be better off by unilaterally altering his travel decisions. The state is characterized by the following equilibrium conditions which must hold for every O/D pair $w \in W$ and every path $p \in P_w$:

$$C_p(x^*) - \lambda_w^* \begin{cases} = 0, & \text{if} \quad x_p^* > 0, \\ \geq 0, & \text{if} \quad x_p^* = 0, \end{cases} \tag{21}$$

and

$$\sum_{p \in P_w} x_p^* \begin{cases} = d_w(\lambda^*), & \text{if} \quad \lambda_w^* > 0, \\ \geq d_w(\lambda^*), & \text{if} \quad \lambda_w^* = 0. \end{cases} \tag{22}$$

Condition (21) states that all utilized paths connecting an O/D pair have equal and minimal travel costs which are equal to the travel disutility associated with traveling between that O/D pair. Condition (22) states that the market clears for each O/D pair under a positive price or travel disutility. As described in Dafermos and Nagurney (1984b) the transportation network equilibrium conditions (21) and (22) can be expressed as the variational inequality: determine $(x^*, \lambda^*) \in R_+^{n_P+n_W}$ such that

$$\sum_{w \in W} \sum_{p \in P_w} [C_p(x^*) - \lambda_w^*] \times \left[x_p - x_p^*\right] + \sum_{w \in W} \sum_{p \in P_w} \left[x_p^* - d_w(\lambda^*)\right] \times [\lambda_w - \lambda_w^*] \geq 0, \quad \forall (x, \lambda) \in R_+^{n_P+n_W}. \tag{23}$$

Note that variational inequality (23) is in *path* flows. Now we also provide the equivalent variational inequality but in link flows, also due to Dafermos and Nagurney (1984b). For additional background, see the book by Nagurney (1999).

Theorem 2

A travel link flow pattern and associated travel demand and disutility pattern is a transportation network equilibrium if and only if it satisfies the variational inequality problem:

determine $(f^*, d^*, \lambda^*) \in \mathcal{K}^5$ *satisfying*

$$\sum_{a\in L} c_a(f^*)\times(f_a-f_a^*)-\sum_{w\in W}\lambda_w^*\times(d_w-d_w^*)+\sum_{w\in W}[d_w^*-d_w(\lambda^*)]\times[\lambda_w-\lambda_w^*]\geq 0, \quad \forall(f,d,\lambda)\in\mathcal{K}^5, \tag{24}$$

where $\mathcal{K}^5 \equiv \{(f, d, \lambda) \in R_+^{K+2n_W} \,|$ *there exists an* x *satisfying* (16) *and* $d_w = \sum_{p\in P_w} x_p, \forall w\}$.

In the next section, we will reformulate the multiperiod supply chain network model in Section 2 as a properly configured transportation network (through a supernetwork construction) by showing that the link flow variational inequality (24) for the constructed transportation network coincides with variational inequality (16).

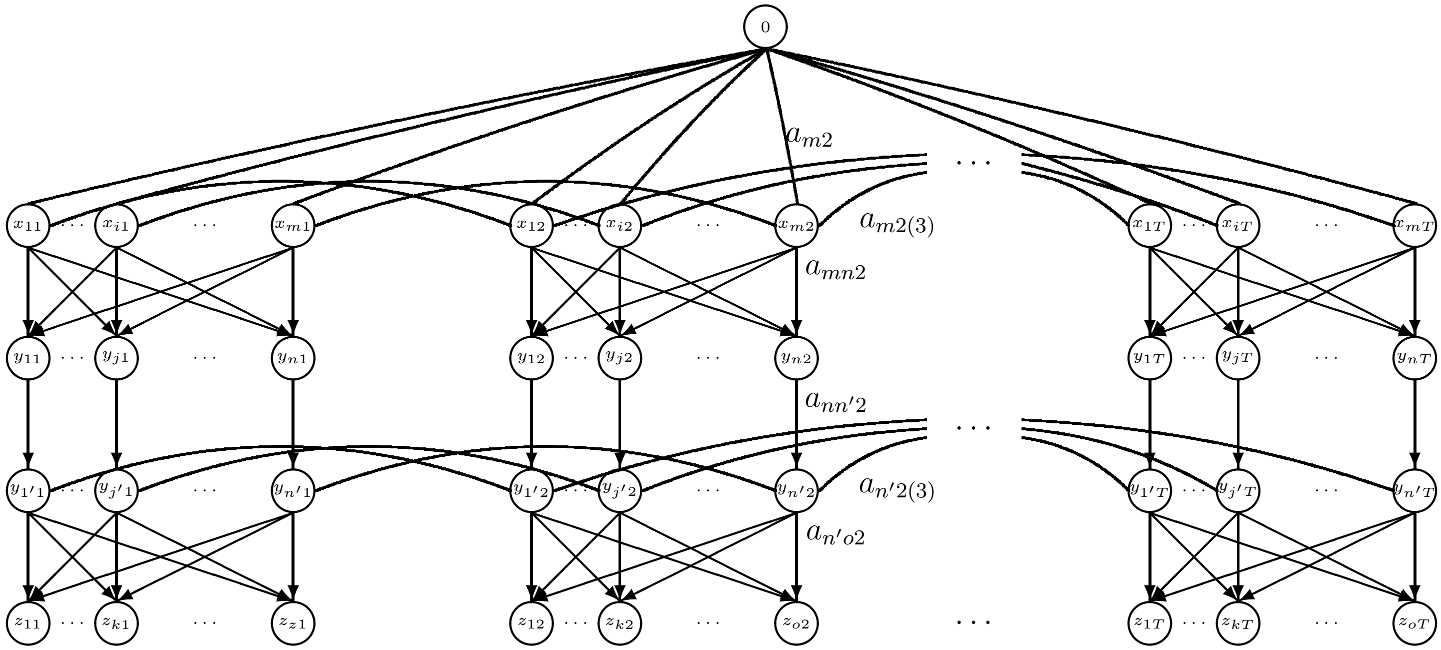

Figure 2: The $\mathcal{G}_S$ Supernetwork Representation of the Multiperiod Supply Chain Network

4. Transportation Network Equilibrium Reformulation of Multiperiod Supply Chain Network Equilibrium

In this section, we establish the supernetwork equivalence of the multiperiod supply chain network equilibrium with a properly configured transportation network equilibrium model with elastic demand as discussed in Section 3.

We consider a multiperiod supply chain network as discussed in Section 2 which consists of T periods; m manufacturers: $i = 1, \ldots, m$; n retailers: $j = 1, \ldots, n$, and o demand markets: $k = 1, \ldots, o$. The supernetwork $\mathcal{G_S}$ of the isomorphic transportation network equilibrium model is depicted in Figure 2 and is constructed as follows. The supernetwork $\mathcal{G_S}$ consists of the single origin node 0 at the top tier, and $o \times T$ destination nodes at the bottom tier denoted, respectively, by z_{kt_3}; $k = 1, \ldots, o$; $t_3 = 1, \ldots, T$. Thus, there are oT O/D pairs in $\mathcal{G_S}$ denoted, respectively, by $w_{11} = (0, z_{11})$, …, $w_{kt_3} = (0, z_{kt_3})$,…, $w_{oT} = (0, z_{oT})$. Node 0 is connected to each second-tiered node x_{it_1}, where $i = 1, \ldots, m$ and $t_1 = 1, \ldots, T$. Each second-tiered node x_{it_1}, in turn, is connected to each third-tiered node y_{jt_2} with $t_2 = t_1$, and $j = 1, \ldots, n$. Each node x_{it_1} is also connected to $x_{i(t_1+1)}$ with the same subscript i.

Each node y_{jt_2}, in turn, is connected with a corresponding node $y_{j't_2}$ in the fourth tier by a single link. Each node $y_{j't_2}$ is linked to node $y_{j't_2+1}$ with the same j' in the same tier by a single link. Finally, from each fourth-tiered node $y_{j't_2}$ there are o links emanating to the bottom-tiered nodes z_{kt_3} with $t_3 = t_2$. There are, hence, $1 + T(m + 2n + o)$ nodes, $K = T(m+mn+n+no)+(T-1)(m+n)$ links, $n_W = oT$ O/D pairs, and $n_P = \frac{(T+1)T}{2}mno$ paths in the supernetwork in Figure 2.

We now define the links in the supernetwork in Figure 2 and the associated flows. Let a_{it_1} denote the link from node 0 to node x_{it_1} with associated link flow $f_{a_{it_1}}$, for $i = 1, \ldots, m$;$t_1 = 1, \ldots, T$. Let $a_{it_1(t_1+1)}$ denote the link from node x_{it_1} to node $x_{i(t_1+1)}$ with associated link flow $f_{a_{it_1(t_1+1)}}$ for $i = 1, \ldots, m$, and $t_1 = 1, \ldots, T-1$. Let a_{ijt_2} denote the link from node x_{it_1} to node y_{jt_2} with associated link flow $f_{a_{ijt_2}}$ for $i = 1, \ldots, m$; $j = 1, \ldots, n$, and $t_1 = t_2 = 1, \ldots, T$. Also, let $a_{jj't_2}$ denote the link connecting node y_{jt_2} with node $y_{j't_2}$ with associated link flow $f_{a_{jj't_2}}$ for j; $j = 1, \ldots, n$, j'; $j' = 1, \ldots, n$, and $t_2 = 1, \ldots, T$. Let $a_{j't_2(t_2+1)}$ denote the link from node $y_{j't_2}$ to node $y_{j'(t_2+1)}$ with associated link flow $f_{a_{j't_2(t_2+1)}}$ for $j' = 1, \ldots, n$ and $t_2 = 1, \ldots, T-1$. Finally, let $a_{j'kt_3}$ denote the link joining node $y_{j't_2}$ with node z_{kt_3} for $j' = 1', \ldots, n'$; $k = 1, \ldots, o$, and $t_2 = t_3 = 1, \ldots, T$, and with associated link flow $f_{a_{j'kt_3}}$.

We group the $\{f_{a_{it_1}}\}$ into the vector f^1; the $\{f_{a_{ijt_2}}\}$ into the vector f^2; the $\{f_{a_{it_1(t_1+1)}}\}$ into the vector f^3; the $\{f_{a_{jj't_2}}\}$ into the vector f^4; the $\{f_{a_{j't_2(t_2+1)}}\}$ into the vector f^5, and the $\{f_{a_{j'kt_3}}\}$ into the vector f^6.

Hence, the paths in $\mathcal{G_S}$, $p_{it_1jt_2j'kt_3}$, can be classified into four groups based on the types of links that they are comprised of. The first group of paths consists of four links: a_{it_1}, a_{ijt_2}, $a_{jj't_2}$, and $a_{j'kt_3}$ with $t_1 = t_2 = t_3$. In this group of paths there are no inventory links. The second group of paths consists of five types of links: a_{it_1}, $a_{it_1(t_1+1)}$, a_{ijt_2}, $a_{jj't_2}$, and $a_{j'kt_3}$ with $t_1 < t_2 = t_3$. In this group of paths there are inventory links at the manufacturers. The third group of paths also consist of five types of links: a_{it_1}, a_{ijt_2}, $a_{jj't_2}$, $a_{j't_2(t_2+1)}$, and $a_{j'kt_3}$ with $t_1 = t_2 < t_3$. In this group there are inventory links at the retailers. Finally, the fourth group of paths consists of six types of links: a_{it_1}, $a_{it_1(t_1+1)}$, a_{ijt_2}, $a_{jj't_2}$, $a_{j't_2(t_2+1)}$, and $a_{j'kt_3}$ with $t_1 < t_2 < t_3$. In this group there are inventory links at the manufacturers and at the retailers.

We denote the path flow associated with path $p_{it_1jt_2j'kt_3}$ by $x_{p_{it_1jt_2j'kt_3}}$. Also, we let $d_{w_kt_3}(\lambda_{w_kt_3})$ denote the known elastic demand function associated with O/D pair w_k at time period t_3 and we let $\lambda_{w_kt_3}$ denote the travel disutility associated with O/D pair w_k at time period t_3.

We assume that the link flows satisfy the conservation of flow equations (17), that is:

$$f_{a_{it_1}} = \sum_{j=1}^{n} \sum_{j'=1'}^{n'} \sum_{k=1}^{o} \sum_{t_2=t_1}^{T} \sum_{t_3=t_2}^{T} x_{p_{it_1jt_2j'kt_3}}, \quad i=1,\ldots,m; t_1=1,\ldots,T, \tag{25}$$

$$f_{a_{ijt_2}} = \sum_{j'=1'}^{n'} \sum_{k=1}^{o} \sum_{t_1=1}^{t_2} \sum_{t_3=t_2}^{T} x_{p_{it_1jt_2j'kt_3}}, \quad i=1,\ldots,m; j=1,\ldots,n; t_2=1,\ldots,T, \tag{26}$$

$$f_{a_{jj't_2}} = \sum_{i=1}^{m} \sum_{k=1}^{o} \sum_{t_1=1}^{t_2} \sum_{t_3=t_2}^{T} x_{p_{it_1jt_2j'kt_3}}, \quad j=1,\ldots,n; j'=1,\ldots,n; t_2=1,\ldots,T, \tag{27}$$

$$f_{a_{j'kt_3}} = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{t_1=1}^{t_3} \sum_{t_2=t_1}^{t_3} x_{p_{it_1jt_2j'kt_3}}, \quad j'=1,\ldots,n; k=1,\ldots,o; t_3=1,\ldots,T, \tag{28}$$

$$f_{a_{it_1(t_1+1)}} = \sum_{j=1}^{n} \sum_{j'=1'}^{n'} \sum_{k=1}^{o} \sum_{t_2=t_1+1}^{T} \sum_{t_3=t_2}^{T} x_{p_{it_1jt_2j'kt_3}}, \quad i=1,\ldots,m; t_1=1,\ldots,T-1, \tag{29}$$

$$f_{a_{j't_2(t_2+1)}} = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{t_1=1}^{t_2} \sum_{t_3=t_2+1}^{T} x_{p_{it_1jt_2j'kt_3}}, j'=1,\ldots,n; t_2=1,\ldots,T-1. \tag{30}$$

Also, we have that

$$d_{w_kt_3} = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{j'=1}^{n'} \sum_{t_1=1}^{t_3} \sum_{t_2=t_1}^{t_3} x_{p_{it_1jt_2j'kt_3}}, \quad k=1,\ldots,o; t_3=1,\ldots,T. \tag{31}$$

A path flow pattern induces a feasible link flow pattern if all path flows are nonnegative and (25)–(31) are satisfied.

Given a feasible product shipment/transaction pattern for the supply chain model with elastic demands, $(q, u^1, Q^1, h, u^2, Q^2) \in \mathcal{K}^4$, we may construct a feasible link flow pattern on the network $\mathcal{G}_{\mathcal{S}}$ as follows: the link flows are defined as:

$$q_{it} \equiv f_{a_{it_1}}, \quad i=1,\ldots,m; t_1=t=1,\ldots,T, \tag{32}$$

$$u_{it} \equiv f_{a_{it_1(t_1+1)}}, \quad i = 1, \ldots, m; t_1 = t = 1, \ldots, T-1, \tag{33}$$

$$q_{ijt} \equiv f_{a_{ijt_2}}, \quad i = 1, \ldots, m; j = 1, \ldots, n; t_2 = t = 1, \ldots, T, \tag{34}$$

$$h_{jt} \equiv f_{a_{jj't_2}}, \quad j = 1, \ldots, n; j' = 1, \ldots, n'; t_2 = t = 1, \ldots, T, \tag{35}$$

$$u_{jt} \equiv f_{a_{j't_2(t_2+1)}}, \quad j = 1, \ldots, n; j' = 1, \ldots, n'; t_2 = t = 1, \ldots, T-1, \tag{36}$$

$$q_{jkt} = f_{a_{j'kt_3}}, \quad j = 1, \ldots, n; j' = 1', \ldots, n'; k = 1, \ldots, o; t_3 = t = 1, \ldots, T. \tag{37}$$

Note that if $(q, u^1, Q^1, h, u^2, Q^2)$ is feasible then the link flow pattern constructed according to (32) – (37) is also feasible and the corresponding path flow pattern that induces such a link flow pattern is, hence, also feasible.

We now assign travel costs on the links of the network $\mathcal{G}_{\mathcal{S}}$ as follows: with each link a_i we assign a travel cost $c_{a_{it_1}}$ defined by

$$c_{a_{it_1}} \equiv \frac{\partial f_{it}(q_t)}{\partial q_{it}}, \quad i = 1, \ldots, m; t_1 = t = 1, \ldots, T; \tag{38}$$

with each link $a_{it(t+1)}$ we assign a travel cost $c_{a_{it_1(t_1+1)}}$ defined by:

$$c_{a_{it_1(t_1+1)}} \equiv \frac{\partial cv_{it}(u_t^1)}{\partial u_{it}}, \quad i = 1, \ldots, m; j = 1, \ldots, n; t_1 = t = 1, \ldots, T-1; \tag{39}$$

with each link a_{ijt_2} we assign a travel cost $c_{a_{ijt_2}}$ defined by:

$$c_{a_{ijt_2}} \equiv \frac{\partial c_{ijt}(q_{ijt})}{\partial q_{ijt}}, \quad i = 1, \ldots, m; j = 1, \ldots, n; t_2 = t = 1, \ldots, T, \tag{40}$$

and with each link jj' we assign a travel cost defined by

$$c_{a_{jj't_2}} \equiv \frac{\partial c_{jt}(h_t)}{\partial h_{jt}}, \quad j = 1, \ldots, n; j' = 1, \ldots, n; t_2 = t = 1, \ldots, T, \tag{41}$$

and with each link $j't_2(t_2+1)$ we assign a travel cost defined by

$$c_{a_{j't_2(t_2+1)}} \equiv \frac{\partial cv_{jt}(u_t^2)}{\partial u_{j't}}, \quad j = 1, \ldots, n; j' = 1, \ldots, n; t_2 = t = 1, \ldots, T-1. \tag{42}$$

Finally, for each link $a_{j'kt_3}$ we assign a travel cost defined by

$$c_{a_{j'kt_3}} \equiv \hat{c}_{jkt}(Q_t^2), \quad j = 1, \ldots, n; j' = 1, \ldots, n'; k = 1, \ldots, o; t_3 = t = 1, \ldots, T. \tag{43}$$

Hence, a traveler traveling on path $p_{it_1jt_2j'kt_3}$ experiences a travel cost $C_{p_{it_1jt_2j'kt_3}}$ given by

$$C_{p_{it_1jt_2j'kt_3}} = c_{a_{it_1}} + \sum_{\tau=t_1}^{t_2-1} c_{a_{i\tau(\tau+1)}} + c_{a_{ijt_2}} + c_{a_{jj't_2}} + \sum_{\tau=t_2}^{t_3-1} c_{a_{j'\tau(\tau+1)}} + c_{a_{j'kt_3}}$$

$$= \frac{\partial f_{it}(q_{it_1})}{\partial q_{it_1}} + \sum_{\tau=t_1}^{t_2-1} \frac{\partial cv_{it}(u^1_\tau)}{\partial u_{i\tau}} + \frac{\partial c_{ijt}(q_{ijt_2})}{\partial q_{ijt_2}} + \frac{\partial c_{jt}(h_{t_2})}{\partial h_{jt_2}} + \sum_{\tau=t_2}^{t_3-1} \frac{\partial cv_{jt}(u^2_\tau)}{\partial u_{j'\tau}} + \hat{c}_{jkt}(Q^2_{t_3}). \quad (44)$$

Also, we define the travel demands associated with the O/D pairs as follows:

$$d_{w_kt_3} \equiv d_{kt}, \quad k = 1, \ldots, o; t_3 = t = 1, \ldots, T, \quad (45)$$

and the travel disutilities:

$$\lambda_{w_{kt_3}} \equiv \rho_{3kt}, \quad k = 1, \ldots, o; t_3 = t = 1, \ldots, T. \quad (46)$$

Consequently, according to the elastic demand transportation network equilibrium conditions (21) and (22), we have that, for each O/D pair w_{kt_3} in $\mathcal{G_S}$ and every path connecting the O/D pair w_{kt_3}, the following conditions must hold:

$$C_{p_{it_1jt_2j'kt_3}} - \lambda^*_{w_{kt_3}} =$$

$$\frac{\partial f_{it}(q^*_{it_1})}{\partial q_{it_1}} + \sum_{\tau=t_1}^{t_2-1} \frac{\partial cv_{it}(u^{1*}_\tau)}{\partial u_{i\tau}} + \frac{\partial c_{ijt}(q^*_{ijt_2})}{\partial q_{ijt_2}} + \frac{\partial c_{jt}(h^*_{t_2})}{\partial h_{jt_2}} + \sum_{\tau=t_2}^{t_3-1} \frac{\partial cv_{jt}(u^{2*}_\tau)}{\partial u_{j'\tau}} + \hat{c}_{jkt}(Q^{2*}_{t_3}) - \lambda^*_{w_{kt_3}}$$

$$\begin{cases} = 0, & \text{if} \quad x^*_{p_{it_1jt_2j'kt_3}} > 0, \\ \geq 0, & \text{if} \quad x^*_{p_{it_1jt_2j'kt_3}} = 0, \end{cases} \quad (47)$$

and

$$\sum_{p \in P_{w_k}} x^*_{p_{it_1jt_2j'kt_3}} \begin{cases} = d_{w_{kt_3}}(\lambda^*_{wt_3}), & \text{if} \quad \lambda^*_{w_kt_3} > 0, \\ \geq d_{w_{kt_3}}(\lambda^*_{wt_3}), & \text{if} \quad \lambda^*_{w_kt_3} = 0. \end{cases} \quad (48)$$

We now provide the variational inequality formulation of the equilibrium conditions (47) and (48) in link form as in (24). According to Theorem 2, a link flow pattern $f^* \in \mathcal{K}^4$ is an equilibrium according to (47) and (48), if and only if it satisfies:

$$\sum_{t_1=1}^{T} \sum_{i=1}^{m} c_{a_{it_1}}(f^{1*}) \times (f_{a_{it_1}} - f^*_{a_{it_1}}) + \sum_{t_2=1}^{T} \sum_{i=1}^{m} \sum_{j=1}^{n} c_{a_{ijt_2}}(f^{2*}) \times (f_{a_{ijt_2}} - f^*_{a_{ijt_2}})$$

$$+\sum_{t_1=1}^{T-1}\sum_{i=1}^{m} c_{a_{it_1(t_1+1)}}(f^{3*})\times(f_{a_{it_1(t_1+1)}}-f^*_{a_{it_1(t_1+1)}})+\sum_{t_2=1}^{T}\sum_{j=1}^{n}\sum_{j'=1'}^{n'} c_{a_{jj't_2}}(f^{4*})\times(f_{a_{jj't_2}}-f^*_{a_{jj't_2}})$$

$$+\sum_{t_2=1}^{T-1}\sum_{j'=1}^{n'} c_{a_{j't_2(t_2+1)}}(f^{5*})\times(f_{a_{j't_2(t_2+1)}}-f^*_{a_{j't_2(t_2+1)}})+\sum_{t_3=1}^{T}\sum_{j'=1}^{n'}\sum_{k=1}^{n} c_{a_{j'kt_3}}(f^{6*})\times(f_{a_{j'kt_3}}-f^*_{a_{j'kt_3}})$$

$$-\sum_{t_3=1}^{T}\sum_{k=1}^{o}\lambda^*_{w_kt_3}\times(d_{w_kt_3}-d^*_{w_kt_3})+\sum_{t_3=1}^{T}\sum_{k=1}^{o}\Big[d^*_{w_kt_3}-d_{w_kt_3}(\lambda^*_w)\Big]\times\Big[\lambda_{w_kt_3}-\lambda^*_{w_kt_3}\Big]\geq 0,\quad \forall f\in\mathcal{K}^4, \tag{49}$$

which, through expressions (32) – (37), and (38) – (43) yields: determine

$$(q^*,h^*,u^{1*},Q^{1^*},u^{2*},Q^{2^*},d^*,\rho_3^*)\in\mathcal{K}^4$$

satisfying:

$$\sum_{t=1}^{T}\sum_{i=1}^{m}\frac{\partial f_{it}(q_t^*)}{\partial q_{it}}\times[q_{it}-q^*_{it}]+\sum_{t=1}^{T}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\partial c_{ijt}(q^*_{ijt})}{\partial q_{ijt}}\times\Big[q_{ijt}-q^*_{ijt}\Big]+\sum_{t=1}^{T}\sum_{j=1}^{n}\frac{\partial c_{jt}(h_t^*)}{\partial h_{jt}}\times\Big[h_{jt}-h^*_{jt}\Big]$$

$$+\sum_{t=1}^{T}\sum_{i=1}^{m}\frac{\partial cv_{it}(u^*_{it})}{\partial u_{it}}\times[u_{it}-u^*_{it}]+\sum_{t=1}^{T}\sum_{j=1}^{n}\frac{\partial cv_{jt}(u^*_{jt})}{\partial u_{jt}}\times\Big[u_{jt}-u^*_{jt}\Big]$$

$$+\sum_{t=1}^{T}\sum_{j=1}^{n}\sum_{k=1}^{o}\hat{c}_{jkt}(Q_t^{2*})\times\Big[q_{jkt}-q^*_{jkt}\Big]+\sum_{t=1}^{T}\sum_{k=1}^{o}\rho^*_{3kt}\times[d_{kt}-d^*_{kt}]$$

$$+\sum_{t=1}^{T}\sum_{k=1}^{o}[d^*_{kt}-d_{kt}(\rho^*_{3t})]\times[\rho_{3kt}-\rho^*_{3kt}]\geq 0,\quad \forall(q,h,u^1,Q^1,u^2,Q^2,d,\rho_3)\in\mathcal{K}^4. \tag{50}$$

But variational inequality (50) is precisely variational inequality (16) governing the multiperiod supply chain network equilibrium with elastic demands.

Hence, we have the following result:

Theorem 3

A solution $(q^*,h^*,u^{1*},Q^{1^*},u^{2*},Q^{2^*},d^*,\rho_3^*)\in\mathcal{K}^4$ *of the variational inequality (16) governing the multiperiod supply chain network equilibrium coincides with the (via (32) – (37) and (38) – (43)) feasible link flow for the supernetwork* $\mathcal{G}_{\mathcal{S}}$ *constructed above and satisfies variational inequality (24); equivalently, variational inequality (50). Hence, it is a transportation network equilibrium according to Theorem 2.*

Remark

This supernetwork equivalence provides the multiperiod supply chain network with an interesting interpretation in terms of paths and path flows. A path corresponds to an end-to-end supply chain that may consist of various types of links, and spans not only the space but also the time periods. The links on a path may represent production, handling processes, transportation and inventorying, which may be owned and operated by heterogenous entities. The cooperation between/among these entities makes the path flows feasible for serving the end-customers in the periods. The paths, or the end-to-end supply chains, however, dynamically influence and compete with one another. In the resultant equilibrium, at each period, all the active paths ending at a market have the same and minimum path cost. The paths with cost higher than the minimum are inactive.

It is also worth noting that, in the case of a single period, the model collapses to a variant of the Nagurney et al. (2002) supply chain network equilibrium model, which was transformed into a transportation network equilibrium problem by Nagurney (2006a).

This modeling framework offers great modeling flexibility. For example, perishable products with a fixed lifetime $\mathcal{L}$ can be easily handled by removing all the paths with $t_3 - t_1 > \mathcal{L}$ from the reformulated transportation network. In addition, if there is a transportation delay $\mathcal{E}_{ij}$ between manufacturer i and retailer j, we can simply modify Figure 1 by adding the transaction/transportation links from node it to node $j(t + \mathcal{E}_{ij})$; $t = 1, ..., T - \mathcal{E}_{ij}$ and by removing the links from node it to node jt; $t = 1, ..., T$. One can easily see that the reformulation and computation can be adjusted accordingly without significant effort.

Finally, it is important to emphasize that existence and uniqueness results for the supply chain network model of Section 2 as well as stability and sensitivity analysis results can now be obtained directly through the transportation network equilibrium reformulation using the Theorems in Dafermos and Nagurney (1984b).

6. Multiperiod Supply Chain Network Examples with Computations

In this section, we provide numerical examples to demonstrate how the theoretical results in this paper can be applied in practice. The algorithms used for the computations were coded in Matlab using a Dell laptop computer. We report the solutions in terms of link flows, rather than path flows, due to space limitations.

The General Equilibration Algorithm

In Examples 1 and 2 below, the multiperiod supply chain networks were first reformulated as the isomorphic elastic demand transportation networks. We then inverted the demand functions (since they were separable and this could easily be done), and transformed the elastic demand transportation networks into equivalent fixed demand transportation networks (see Gartner (1982) and Nagurney (1999)). The general equilibration algorithm (cf. Dafermos and Sparrow (1969) and Nagurney (1999)) was then applied to compute the solutions of the fixed demand transportation networks. In Examples 1 and 2, for the reformulated fixed demand transportation network problems, the initial value of the path that directly connected the origin and each destination was set equal to the fixed demand of that O/D pair. The initial values of all the other paths were equal to zero. For completeness and easy reference, we now state the general equilibration algorithm.

Single O/D Pair Equilibration

Step 0: Initialization

Construct an initial feasible flow pattern x^0 which induces a feasible link flow pattern. Set $\mathcal{T} := 1$.

Step 1: Selection and Convergence Verification

Determine

$$r = \{p| \max_p \; C_p \text{ and } x_p^{\mathcal{T}-1} > 0\} \tag{51}$$

$$q = \{p| \min_p \; C_p\}. \tag{52}$$

If $|C_r - C_q| \leq \epsilon$, with $\epsilon > 0$, then stop; otherwise, go to step 2.

Step 2: Computation

Compute

$$\Delta' = \frac{|C_r - C_q|}{\sum_{\alpha \in L} g_a(\delta_{\alpha q} - \delta_{\alpha r})^2}, \tag{53}$$

$$\Delta = \min(\Delta', x_r^{\mathcal{T}-1}). \tag{54}$$

Set

$$x_r^{\mathcal{T}} = x_r^{\mathcal{T}-1} - \Delta \tag{55}$$

$$x_q^{\mathcal{T}} = x_q^{\mathcal{T}-1} + \Delta \tag{56}$$

$$x_p^{\mathcal{T}} = x_p^{\mathcal{T}-1}, \forall p \neq q \cup r. \tag{57}$$

Let $\mathcal{T} := \mathcal{T} + 1$, and go to Step 1.

On a network in which there are multiple O/D pairs, we first term Step 1 (without convergence check) and Step 2 above as the equilibration operator E_{w_i} for O/D pair w_i, and then proceed as follows.

Let

$$E \equiv E_{w_Z} \circ \ldots \circ E_{w_1}. \tag{58}$$

Operator E sequentially applies $E_{w_i}, i = 1, \ldots, Z$.

Step 0: Initialization

Construct an initial feasible flow pattern x^0 which induces a feasible link flow pattern. Set $\mathcal{T} := 1$.

Step 1: Equilibration

Apply E.

Step 2: Convergence Verification

If convergence holds, stop; otherwise, set $\mathcal{T} := \mathcal{T} + 1$, and go to Step 1.

Example 1

The first example consisted of two manufacturers, two retailers, two demand markets, and three time periods, as depicted in Figure 3. Hence, in the first numerical example (see Figure 4), we had that: $T = 3$; $m = 2$; $n = 2$; $n' = 2'$, and $o = 2$.

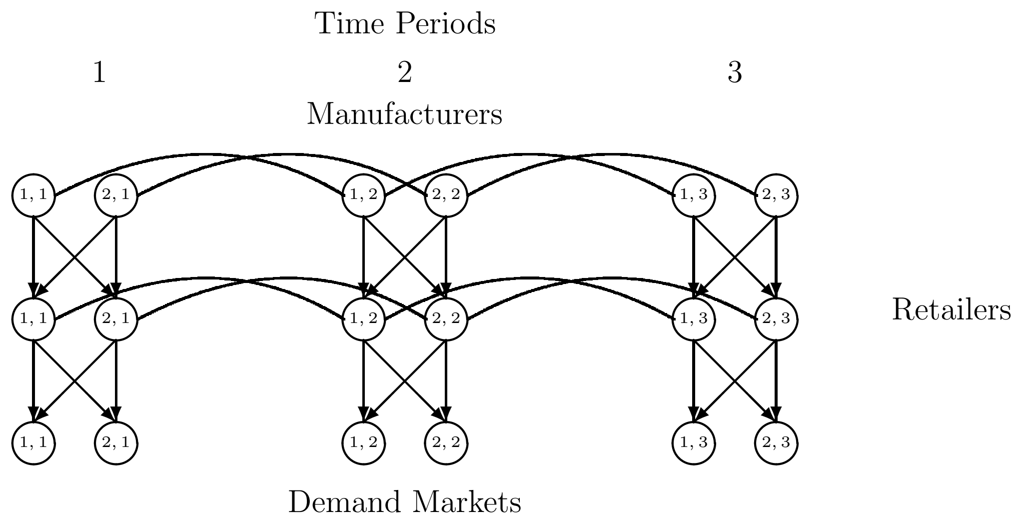

Figure 3: Multiperiod Supply Chain Network for Examples 1, 2, and 3

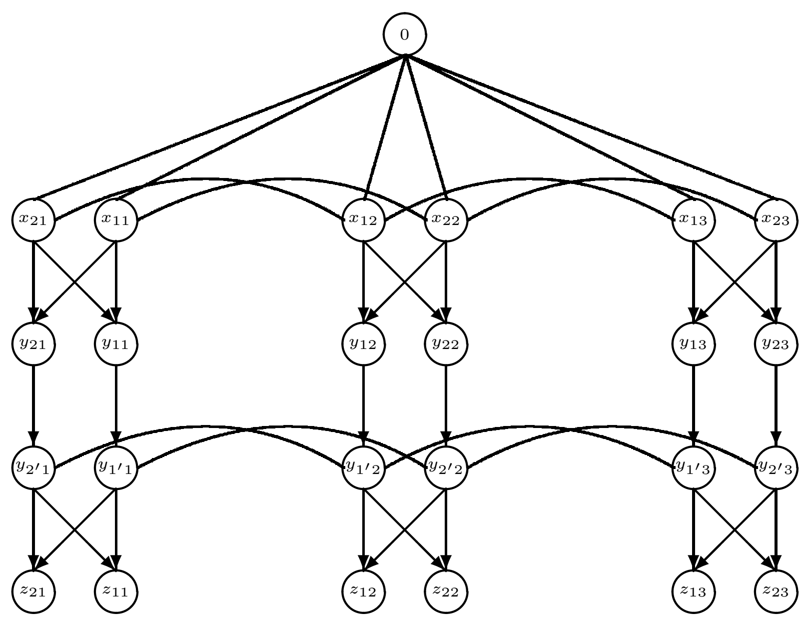

Figure 4: Supernetwork Structure of the Transportation Network Equilibrium Reformulation of Examples 1, 2, and 3

The production cost functions of the manufacturers were given by:

$$f_{1t}(q_{1t}) = 0.1q_{1t}^2 + 5q_{1t} + 20, \quad t = 1,2,3, \quad f_{2t}(q_{2t}) = 0.15q_{2t}^2 + 4q_{2t} + 10, \quad t = 1,2,3.$$

The transaction/transportation cost functions faced by the manufacturers and associated with transacting with the retailers were:

$$c_{11t}(q_{11t}) = 0.005q_{11t}^2 + q_{11t}, \quad c_{12t}(q_{12t}) = 0.005q_{12t}^2 + q_{12t},$$

$$c_{21t}(q_{21t}) = 0.005q_{21t}^2 + q_{21t}, \quad c_{22t}(q_{22t}) = 0.005q_{22t}^2 + q_{22t}, \quad t = 1,2,3.$$

The handling costs of the retailers, in turn, were:

$$c_{1t}(h_{1t}) = 0.05h_{1t}^2 + 1.5h_{1t} + 20, \quad t = 1,2,3, \quad c_{2t}(h_{2t}) = 0.1h_{2t}^2 + h_{2t} + 30, \quad t = 1,2,3.$$

The inventory cost functions of the manufacturers were given by:

$$cv_{1t}(u_{1t}) = 0.025u_{1t}^2 + 0.5u_{1t} + 10, \quad t = 1,2, \quad cv_{2t}(u_{2t}) = 0.025u_{2t}^2 + 0.5u_{2t} + 20, \quad t = 1,2.$$

The inventory costs functions of the retailers were given by:

$$cv_{1t}(u_{1t}) = 0.025u_{1t}^2 + 0.5u_{1t}, \quad t = 1,2, \quad cv_{2t}(u_{2t}) = 0.025u_{2t}^2 + 0.5u_{2t}, \quad t = 1,2.$$

Note that in the above cost functions, since the parameters of the quadratic terms were much smaller than those of the linear terms, the costs were approximately linear when the product flows were low and quadratic when the flows were high.

The unit transaction costs associated with transacting between the retailers and the demand markets were:

$$c_{1kt} = 2, \quad k = 1,2; t = 1,2,3, \quad c_{2kt} = 1, \quad k = 1,2; t = 1,2,3.$$

The demand functions were given by:

$$d_{11}(\rho_{311}) = 15 - \frac{1}{2}\rho_{311}, \quad d_{12}(\rho_{312}) = 20 - \frac{1}{3}\rho_{312}, \quad d_{13}(\rho_{313}) = 80 - \frac{1}{10}\rho_{313},$$

$$d_{21}(\rho_{321}) = 10 - \frac{1}{2}\rho_{321}, \quad d_{22}(\rho_{322}) = 15 - \frac{1}{3}\rho_{322}, \quad d_{23}(\rho_{323}) = 90 - \frac{1}{10}\rho_{323}.$$

Note that the demands at both markets become less price-sensitive in the third time period.

The general equilibration method was utilized to compute the solution of the transportation network which was then translated into the equilibrium solution of the dynamic supply chain network as discussed in Section 3. The algorithm converged after 849 iterations. The equilibrium prices at the manufacturers, ρ^*_{1ijt}; $i = 1, 2$; $j = 1, 2$; $t = 1, 2, 3$, and at the retailers, ρ^*_{2jt}; $j = 1, 2$; $t = 1, 2, 3$ were recovered from the above solution (see Appendix 2 for the computation).

The equilibrium solution for the transportation network equilibrium reformulation as well as the translation into the supply chain network equilibrium solution are given in Table 3. In this example, there was inventorying at both manufacturers and at both retailers in time periods 1 and 2.

Example 2

The second example had the same data as Example 1, except that the product was now assumed to be perishable with a lifetime $\mathcal{L} = 2$. Thus, in this example, the products produced in the first period were not allowed to be in the market in the third period.

The multiperiod supply chain network problem was reformulated as a transportation network equilibrium model where the paths with $t_3 - t_1 > 2$ were eliminated. The equilibrium flow pattern was computed using the general equilibration method which converged after 573 iterations. We then translated the solution to the multiperiod supply chain flows and prices, as reported in Table 3. The equilibrium prices, ρ^*_{1ijt}; $i = 1, 2$; $j = 1, 2$: $t = 1, 2, 3$, and ρ^*_{2jt}; $j = 1, 2$; $t = 1, 2, 3$ were also recovered and are shown in Table 3. In this example, there was no inventorying of the product at the manufacturers from the first to the second time period.

We now further discuss and compare the results for Examples 1 and 2. First, since the demand for the product spiked and then became less price-sensitive in period 3, the prices at the demand markets in the third period increased to the highest level in both examples. Also, as we can expect, the market price of the perishable product was more volatile than that of the nonperishable product. Moreover, it was a little counter-intuitive to observe that for the perishable product, in the second period, the retailers paid \$15.2808 per unit to obtain the product from the manufacturers while selling the product to the consumers at \$14.5485 per unit. This interesting result actually make sense in this example, because in the second period, the product sold by the retailers was produced in period 1 and would expire before period 3, whereas the product that they purchased was produced in period 2 and could be sold at a high price of \$23.7515 at period 3.

Table 3: Equilibrium Solutions of Examples 1 and 2

		Example 1			Example 2	
Variable	$t=1$	$t=2$	$t=3$	$t=1$	$t=2$	$t=3$
$f^*_{a_{1t}} = q^*_{1t}$	32.6997	36.3997	40.5709	18.5877	45.5601	49.1646
$f^*_{a_{2t}} = q^*_{2t}$	25.2589	27.7443	30.5455	15.7721	33.9027	36.3221
$f^*_{a_{11t}} = q^*_{11t}$	16.6645	21.3144	29.6492	11.2588	24.2616	33.5997
$f^*_{a_{12t}} = q^*_{12t}$	11.2351	13.2002	17.6067	7.3289	16.8807	19.9826
$f^*_{a_{21t}} = q^*_{21t}$	12.8955	16.9915	24.6881	9.8501	18.3844	27.2268
$f^*_{a_{22t}} = q^*_{22t}$	7.4509	8.8588	12.6637	5.9219	11.0021	13.6114
$f^*_{a_{11't}} = h^*_{1t}$	29.5601	38.3060	54.3373	21.1089	42.6460	60.8265
$f^*_{a_{22't}} = h^*_{2t}$	18.6860	22.0591	30.2705	13.2509	27.8829	33.5940
$f^*_{a_{1'1t}} = q^*_{11t}$	5.4186	12.8893	38.7451	6.4787	5.2527	46.0004
$f^*_{a_{1'2t}} = q^*_{12t}$	0.9186	8.2277	56.0039	2.4797	6.8976	57.4722
$f^*_{a_{2'1t}} = q^*_{21t}$	0.0000	0.1694	38.9204	0.8006	9.2310	31.4244
$f^*_{a_{2'2t}} = q^*_{22t}$	0.0000	0.1643	31.7615	0.2996	2.9194	30.0525
$f^*_{a_{1t(t+1)}} = u^*_{1t}$	4.8001	6.6850		0.0000	4.4177	
$f^*_{a_{2t(t+1)}} = u_{2t}$	4.9124	6.8063		0.0000	4.5162	
$f^*_{a_{1't(t+1)}} = u^*_{1t}$	23.2227	40.4117		12.1504	42.6460	
$f^*_{a_{2't(t+1)}} = u^*_{2t}$	18.6860	40.4114		12.1505	27.8829	
$d^*_{w_1t} = d^*_{1t}$	5.4186	13.0587	77.6655	7.2794	14.4838	77.4248
$d^*_{w_2t} = d^*_{2t}$	0.9186	8.3920	87.7655	2.7794	9.8171	87.5248
ρ^*_{111t}	12.7065	13.4931	14.4106	9.8301	15.3546	16.1689
ρ^*_{121t}	12.7065	13.4931	14.4106	9.8301	15.3546	16.1689
ρ^*_{112t}	12.6523	13.4119	14.2903	9.7908	15.2808	16.0327
ρ^*_{122t}	12.6523	13.4118	14.2903	9.7908	15.2808	16.0327
ρ^*_{21t}	17.1626	18.8237	21.3444	13.4410	14.5485	23.7515
ρ^*_{22t}	17.4438	18.8237	21.3444	13.4410	14.5485	23.7515
$\lambda^*_{w_1t} = \rho^*_{31t}$	19.1626	20.8237	23.3443	15.4410	16.5485	25.7515
$\lambda^*_{w_2t} = \rho^*_{32t}$	18.1626	19.8237	22.3443	14.4410	15.5485	24.7515

The Euler Method

In Examples 3 and 4, the multiperiod supply chain network problems were first reformulated as equivalent transportation network equilibrium problems, over the appropriately constructed supernetworks, as discussed in Section 3. They were then solved using the Euler method which was originally developed for the computation of transportation equilibria (cf. Nagurney and Zhang (1996) and the references therein). For completeness and easy reference, we now present the Euler method for the solution of variational inequality (23), in standard form (cf. Nagurney (1999)), given by: determine $X^* \in \mathcal{K}$ satisfying:

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K},$$

where $X \equiv (x, \lambda)$, $F(X) \equiv (C(x) - B^T\lambda, Bx - d(\lambda))$ where B is the $n_W \times n_P$-dimensional matrix with element $(w, p) = 1$, if path $p \in P_w$, and 0, otherwise, and $\mathcal{K} \equiv \{X | X \in R_+^{n_P + n_W}\}$.

Let $\mathcal{T}$ denote an iteration counter.

Step 0: Initialization

Set $X^0 \in \mathcal{K}$.

Let $\mathcal{T} = 1$ and set the sequence $\{\alpha_\mathcal{T}\}$ so that $\sum_{\mathcal{T}=1}^{\infty} \alpha_\mathcal{T} = \infty$, $\alpha_\mathcal{T} > 0$ for all $\mathcal{T}$, and $\alpha_\mathcal{T} \to 0$ as $\mathcal{T} \to \infty$.

Step 1: Computation

Compute $X^\mathcal{T} \in \mathcal{K}$ by solving the variational inequality subproblem:

$$\langle X^\mathcal{T} + \alpha_\mathcal{T} F(X^{\mathcal{T}-1}) - X^{\mathcal{T}-1}, X - X^\mathcal{T} \rangle \geq 0, \qquad \forall X \in \mathcal{K}. \tag{59}$$

Step 2: Convergence Verification

If $|X^\mathcal{T} - X^{\mathcal{T}-1}| \leq \epsilon$, with $\epsilon > 0$, a pre-specified tolerance, then stop; otherwise, set $\mathcal{T} := \mathcal{T} + 1$, and go to Step 1.

The sequence $\{\alpha_\mathcal{T}\} = \{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots\}$ in the numerical examples. The convergence tolerance ϵ was set to .0001.

Example 3

The third example had the same network structure as Examples 1 and 2. The cost functions, however, were now nonseparable. The production cost functions of the manufacturers were given by:

$$f_{1t}(q_{1t}, q_{2t}) = 5 + q_{1t} + 0.4q_{1t}^2 + 0.2q_{1t}q_{2t}, \quad t = 1, 2, 3,$$

$$f_{2t}(q_{1t}, q_{2t}) = 10 + 2q_{2t} + 1.5q_{2t}^2 + 0.4q_{1t}q_{2t}, \quad t = 1, 2, 3.$$

The transaction/transportation cost functions faced by the manufacturer and associated with transacting with the retailers were:

$$c_{11t}(q_{11t}) = 3q_{11t} + 0.1q_{11t}^2, \quad c_{12t}(q_{12t}) = 3q_{12t} + 0.1q_{12t}^2,$$

$$c_{21t}(q_{21t}) = q_{21t} + 0.1q_{21t}^2, \quad c_{22t}(q_{22t}) = q_{22t} + 0.1q_{22t}^2, \quad t = 1, 2, 3.$$

The handling costs of the retailers, in turn, were:

$$c_{1t}(h_{1t}, h_{2t}) = 0.1h_{1t}^2 + 0.1h_{1t}h_{2t}, \quad t = 1, 2, 3, \quad c_{2t}(h_{1t}, h_{2t}) = 0.1h_{2t}^2 + 0.1h_{1t}h_{2t}, \quad t = 1, 2, 3.$$

The inventory cost functions of the manufacturers were:

$$cv_{1t}(u_{1t}) = 0.2u_{1t}^2 + u_{1t}, \quad t = 1, 2, \quad cv_{2t}(u_{2t}) = 0.5u_{2t}^2 + 2u_{2t}, \quad t = 1, 2.$$

The inventory costs functions of the retailers were:

$$cv_{1t}(u_{1t}) = 0.05u_{1t}^2 + u_{1t}, \quad t = 1, 2, \quad cv_{2t}(u_{2t}) = 0.02u_{2t}^2 + u_{2t}, \quad t = 1, 2.$$

The unit transaction costs associated with transacting between the retailers and the demand market were:

$$c_{jkt} = 1, \quad j = 1, 2; k = 1, 2; t = 1, 2, 3.$$

The demand functions were given by:

$$d_{11}(\rho_{311}) = 70 - \rho_{311}, \quad d_{12}(\rho_{312}) = 80 - \rho_{312}, \quad d_{13}(\rho_{313}) = 90 - \rho_{313},$$

$$d_{21}(\rho_{321}) = 55 - 0.2\rho_{321}, \quad d_{22}(\rho_{322}) = 55 - 0.2\rho_{322}, \quad d_{23}(\rho_{323}) = 60 - 0.2\rho_{323}.$$

This multiperiod supply chain network problem was reformulated as a transportation network equilibrium problem with the supernetwork structure as depicted in Figure 4. We initialized the algorithm by setting all flows equal to zero. The Euler method converged in $1,803$ iterations and computed the transportation network equilibrium flow and price pattern in Table 4, which was then translated into the equilibrium product flows and prices of the multiperiod supply chain network, also given in Table 4. In Example 3, cf. Table 4, there were no inventories held at the manufacturers.

Example 4

The last example had the same data as Example 3 except that we assumed that it now took one period of time to transport the product from manufacturer 1 to the retailers. We adjusted the multiperiod supply chain network in Figure 3 by first removing all the transaction links from manufacturer 1 and then adding the transaction links from manufacturer 1 at period t to both retailers at period $t+1$; $t = 1, 2$. The modified supply chain network with transportation delay is shown in Figure 5.

The multiperiod supply chain network was transformed into the equivalent transportation network equilibrium problem as depicted in Figure 6. We initialized the problem by setting all flows equal to zero, and solved it using the Euler method, with convergence attained after $1,048$ iterations. The equilibrium solution, in transportation and supply chain notation, is given in Table 4. In Example 4, only the retailers had positive amounts of inventory and that was from the second to the last, that is, third, time period.

Note that in Example 4, q^*_{1jt} now denotes the quantity of the product shipped from manufacturer 1 in period t to retailer j in period $t+1$.

In Example 3, since the first manufacturer had a lower production cost, he took most of the market share in the wholesale market. In Example 4, however, because of the transportation delay, the product produced by the first manufacturer could not arrive at the retailers in period 1, which led to a dramatic rise in the price at the demand markets in period 1.

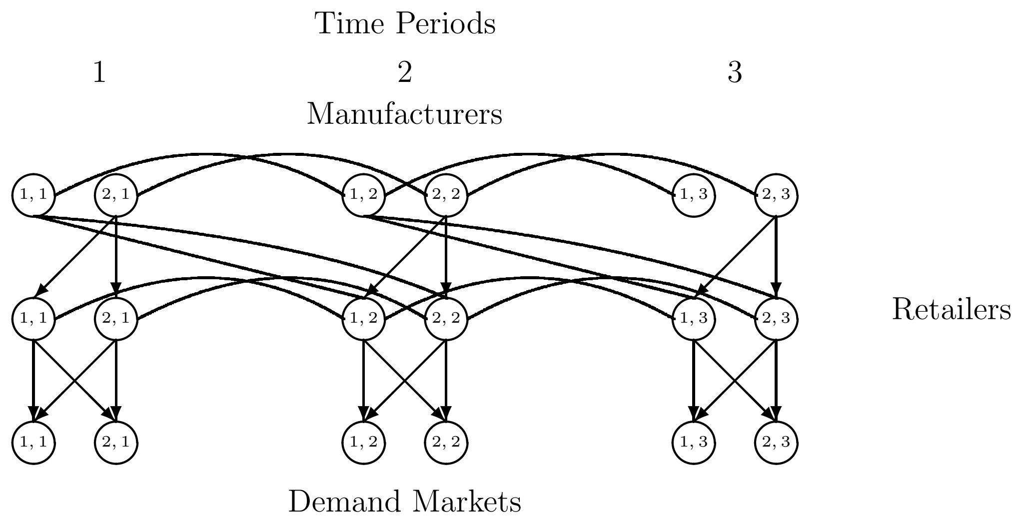

Figure 5: Multiperiod Supply Chain Network for Example 4

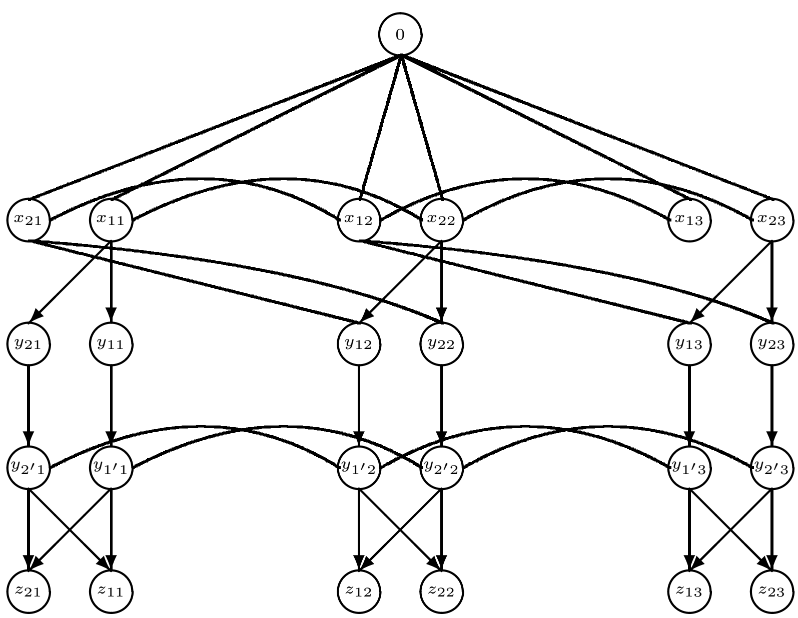

Figure 6: Supernetwork Structure of the Transportation Network Equilibrium Reformulation of Example 4

Table 4: Equilibrium Solutions of Examples 3 and 4

		Example 3			Example 4	
Variable	$t=1$	$t=2$	$t=3$	$t=1$	$t=2$	$t=3$
$f^*_{a_{1t}} = q^*_{1t}$	50.8291	51.9564	53.1257	49.4331	53.4121	0.0000
$f^*_{a_{2t}} = q^*_{2t}$	9.1084	9.3028	9.5044	30.4517	9.7468	16.4581
$f^*_{a_{11t}} = q^*_{11t}$	25.4145	25.9782	26.5628	24.7165	26.7060	0.0000
$f^*_{a_{12t}} = q^*_{12t}$	25.4145	25.9782	26.5628	24.7165	26.7060	0.0000
$f^*_{a_{21t}} = q^*_{21t}$	4.5542	4.6514	4.7522	15.2258	4.8734	8.2290
$f^*_{a_{22t}} = q^*_{22t}$	4.5542	4.6514	4.7522	15.2258	4.8734	8.2290
$f^*_{a_{11't}} = h^*_{1t}$	29.9688	30.6296	31.3150	15.2258	29.5900	34.9351
$f^*_{a_{22't}} = h^*_{2t}$	29.9688	30.6296	31.3150	15.2258	29.5900	34.9351
$f^*_{a_{1'1t}} = q^*_{11t}$	2.3862	6.3799	14.7593	0.0000	7.6204	12.3561
$f^*_{a_{1'2t}} = q^*_{12t}$	21.2911	23.0842	24.0125	15.2258	20.8777	23.6708
$f^*_{a_{2'1t}} = q^*_{21t}$	6.0551	10.8097	11.1321	0.0000	7.9223	12.1430
$f^*_{a_{2'2t}} = q^*_{22t}$	21.3970	19.3536	23.1657	15.2258	21.2307	23.2289
$f^*_{a_{1t(t+1)}} = u^*_{1t}$	0.0000	0.0000		0.0000	0.0000	
$f^*_{a_{2t(t+1)}} = u^*_{2t}$	0.0000	0.0000		0.0000	0.0000	
$f^*_{a_{1't(t+1)}} = u^*_{1t}$	6.2914	7.4568		0.0000	1.0917	
$f^*_{a_{2't(t+1)}} = u^*_{2t}$	2.5165	2.9827		0.0000	0.4368	
$d^*_{w_1t} = d^*_{1t}$	8.4413	17.1897	25.8914	0.0000	15.5428	24.4991
$d^*_{w_2t} = d^*_{2t}$	42.6882	42.4379	46.3314	30.4517	42.1085	46.8998
ρ^*_{111t}	51.5679	52.6213	53.7140	54.5801	54.0203	
ρ^*_{121t}	51.5679	52.6213	53.7140	54.5801	54.0203	
ρ^*_{112t}	51.5679	52.6213	53.7140	117.1736	54.5801	54.0203
ρ^*_{122t}	51.5679	52.6213	53.7140	117.1736	54.5801	54.0203
ρ^*_{21t}	60.5586	61.8102	63.1085	121.7413	63.4571	64.5008
ρ^*_{22t}	60.5586	61.8102	63.1085	121.7413	63.4571	64.5008
$\lambda^*_{w_1t} = \rho^*_{31t}$	61.5586	62.8102	64.1085	70.0000	64.4571	65.5008
$\lambda^*_{w_2t} = \rho^*_{32t}$	61.5586	62.8102	64.1085	122.7413	64.4571	65.5008

6. Summary and Conclusions

In this paper, we developed a multiperiod competitive supply chain network equilibrium model and demonstrated that it could be reformulated and solved as a transportation network equilibrium problem over a properly constructed abstract network or supernetwork. We assumed that the decision-makers in the multiperiod supply chain network had sufficient information of the future and sought the optimal plans that maximized their profits over the planning horizon. At the equilibrium, the prices at each period were mutually determined, and the optimality conditions of all the decision-makers held simultaneously so that no decision-maker could be better off by altering his decisions. This model allowed us to study the interplay of noncooperative decision-makers in a dynamic setting, and to compute the resultant equilibrium pattern of the prices, transactions, and inventories in the multiperiod supply chain network.

The supernetwork equivalence of the multiperiod supply chain network model provides an interesting interpretation of the equilibrium conditions in terms of paths and path flows. This supernetwork equivalence also allowed us to transfer some of the analytical and computational tools developed for transportation networks to the study of multiperiod supply chain equilibrium problems.

In addition, we discussed how this framework could be used to capture both perishability of products and well as time delays associated with transportation through the appropriate changes in the underlying network topologies.

Acknowledgments

This research was supported, in part, by NSF Grant No. IIS-0002647 and, in part, by the John F. Smith Memorial Fund at the Isenberg School of Management. The authors are very grateful for this support.

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Appendix 1: Proof of Theorem 1

We first demonstrate that an equilibrium pattern according to Definition 1 satisfies the variational inequality (16). We sum up inequalities (5), (11), and (15), and, after algebraic simplifications, obtain (16).

We now prove the converse, that is, a solution to variational inequality (16) satisfies the sum of conditions (5), (11), and (15), and is, therefore, a supply chain network equilibrium pattern according to Definition 1.

First, we add the term $\rho^*_{1ijt} - \rho^*_{1ijt}$ to the first term in the second summand expression in (16). Then, we add the term $\rho^*_{2jt} - \rho^*_{2jt}$ to the first term in the sicth summand expression in (16). Because these terms are all equal to zero, they do not change (16) and we obtain the following inequality: determine $(q^*, h^*, u^{1*}, Q^{1^*}, u^{2*}, Q^{2^*}, d^*, \rho^*_3) \in \mathcal{K}^4$ satisfying:

$$\sum_{t=1}^{T}\sum_{i=1}^{m}\frac{\partial f_{it}(q^*_t)}{\partial q_{it}} \times [q_{it} - q^*_{it}] + \sum_{t=1}^{T}\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial c_{ijt}(q^*_{ijt})}{\partial q_{ijt}} + \rho^*_{1ijt} - \rho^*_{1ijt}\right] \times \left[q_{ijt} - q^*_{ijt}\right]$$

$$+\sum_{t=1}^{T}\sum_{j=1}^{n}\frac{\partial c_{jt}(h^*_t)}{\partial h_{jt}} \times \left[h_{jt} - h^*_{jt}\right]$$

$$+\sum_{t=1}^{T}\sum_{i=1}^{m}\frac{\partial cv_{it}(u^*_{it})}{\partial u_{it}} \times [u_{it} - u^*_{it}] + \sum_{t=1}^{T}\sum_{j=1}^{n}\frac{\partial cv_{jt}(u^*_{jt})}{\partial u_{jt}} \times \left[u_{jt} - u^*_{jt}\right]$$

$$+\sum_{t=1}^{T}\sum_{j=1}^{n}\sum_{k=1}^{o}\left[\hat{c}_{jkt}(Q^{2*}_t) + \rho^*_{2jt} - \rho^*_{2jt}\right] \times \left[q_{jkt} - q^*_{jkt}\right] + \sum_{t=1}^{T}\sum_{k=1}^{o}\rho^*_{3kt} \times [d_{kt} - d^*_{kt}]$$

$$+\sum_{t=1}^{T}\sum_{k=1}^{o}[d^*_{kt} - d_{kt}(\rho^*_{3t})] \times [\rho_{3kt} - \rho^*_{3kt}] \geq 0, \quad \forall (q, h, u^1, Q^1, u^2, Q^2, d, \rho_3) \in \mathcal{K}^4, \tag{60}$$

which, in turn, can be rewritten as: determine $(q^*, h^*, u^{1*}, Q^{1^*}, u^{2*}, Q^{2^*}, d^*, \rho_3^*) \in \mathcal{K}^4$ satisfying:

$$\sum_{t=1}^{T}\sum_{i=1}^{m}\frac{\partial f_{it}(q_t^*)}{\partial q_{it}}\times[q_{it}-q_{it}^*]+\sum_{t=1}^{T}\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial c_{ijt}(q_{ijt}^*)}{\partial q_{ijt}}-\rho_{1ijt}^*\right]\times\left[q_{ijt}-q_{ijt}^*\right]+\sum_{t=1}^{T}\sum_{i=1}^{m}\frac{\partial cv_{it}(u_{it}^*)}{\partial u_{it}}\times[u_{it}-u_{it}^*]$$

$$+\sum_{t=1}^{T}\sum_{j=1}^{n}\frac{\partial c_{jt}(h_t^*)}{\partial q_{ijt}}\times\left[h_{jt}-h_{jt}^*\right]+\sum_{t=1}^{T}\sum_{i=1}^{m}\sum_{j=1}^{n}\rho_{1ijt}^*\times\left[q_{ijt}-q_{ijt}^*\right]-\sum_{t=1}^{T}\sum_{j=1}^{n}\sum_{k=1}^{o}\rho_{2jt}^*\times\left[q_{jkt}-q_{jkt}^*\right]$$

$$+\sum_{t=1}^{T}\sum_{j=1}^{n}\frac{\partial cv_{jt}(u_{jt}^*)}{\partial u_{jt}}\times\left[u_{jt}-u_{jt}^*\right]$$

$$+\sum_{t=1}^{T}\sum_{j=1}^{n}\sum_{k=1}^{o}\hat{c}_{jkt}(Q_t^{2*})\times\left[q_{jkt}-q_{jkt}^*\right]+\sum_{t=1}^{T}\sum_{k=1}^{o}\rho_{3kt}^*\times[d_{kt}-d_{kt}^*]$$

$$+\sum_{t=1}^{T}\sum_{k=1}^{o}[d_{kt}^*-d_{kt}(\rho_{3t}^*)]\times[\rho_{3kt}-\rho_{3kt}^*]\geq 0,\quad \forall(q,h,u^1,Q^1,u^2,Q^2,d,\rho_3)\in\mathcal{K}^4. \qquad (61)$$

Clearly, (61) is equal to the sum of the optimality conditions (5) and (11), and the equilibrium conditions (15) and is, hence, a variational inequality governing the supply chain network equilibrium according to Definition 1. □

Appendix 2: Computation of the Equilibrium Prices at the Manufacturers and the Retailers

We discuss how to recover the equilibrium prices ρ_{2jt}^*; $j = 1, \ldots, n$; $t = 1, \ldots, T$, and ρ_{1ijt}^*; $i = 1, \ldots, m$; $j = 1, \ldots, n$, and $t = 1, \ldots, T$ from the solution of variational inequality (16). First, from (12), it follows that if $q_{jkt}^* > 0$, then $\rho_{2jt}^* = \rho_{3kt}^* - \hat{c}_{jkt}(Q_t^{2*})$. In order to retrieve ρ_{1ijt}^*, however, we need to substitute h_{jt} using equation (7), and rewrite variational inequality (11) using the Lagrange multipliers as follows: determine $(Q^{1*}, u^{2*}, Q^{2*}, \gamma_{2jt}^*) \in \mathcal{K}^2$ satisfying:

$$\sum_{t=1}^{T}\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial c_{jt}(Q^{1*})}{\partial q_{ijt}}+\rho_{1ijt}^*-\gamma_{2jt}^*\right]\times\left[q_{ijt}-q_{ijt}^*\right]+\sum_{t=1}^{T}\sum_{j=1}^{n}\sum_{k=1}^{o}\left[\gamma_{2jt}^*-\rho_{2jt}^*\right]\times\left[q_{jkt}-q_{jkt}^*\right]$$

$$+\sum_{t=1}^{T-1}\sum_{j=1}^{n}\left[\gamma_{2jt}^*-\gamma_{2j(t+1)}^*+\frac{\partial cv_{jt}(u_{jt}^*)}{\partial u_{jt}}\right]\times\left[u_{jt}-u_{jt}^*\right]\geq 0,\quad \forall(Q^1,u^2,Q^2,\gamma_2)\in R_+^{Tn(m+o+2)}, \qquad (62)$$

where γ_2 is Tn-dimensional vector with components γ_{2jt}; $t = 2, \ldots, T-1$; $j = 1, \ldots, n$, denoting the Lagrange multipliers for constraint (9), γ_{2j1}; $j = 1, \ldots, n$, for constraint (8), and γ_{2jT}; $j = 1, \ldots, n$, for constraint (10).

From the first summand of (62) it follows that if $q^*_{ijt} > 0$, $\rho^*_{1ijt} = \gamma^*_{2jt} - \frac{\partial c_{jt}(Q^{1*})}{\partial q_{ijt}} = \gamma^*_{2jt} - \frac{\partial c_{jt}(h^*_t)}{\partial h_t}$. Thus, we need to retrieve the value of γ^*_{2jt} before recovering ρ^*_{1ijt}. Since $q^*_{ijt} = f^*_{a_{ijt}}$ is greater than zero, according to (17), there must exist a path $\bar{p}$ that has positive path flow and contains link a_{ijt}. Without loss of generality, we denote path $\bar{p}$ by $p_{itj\bar{t}_2\bar{j}'\bar{k}\bar{t}_3}$ with path flow $x_{p_{i\bar{t}_1jtj'\bar{k}\bar{t}_3}} > 0$. Apparently, according to (17), we have $u_{j\tau} \geq x_{p_{i\bar{t}_1jtj'\bar{k}\bar{t}_3}} > 0, t \leq \tau \leq \bar{t}_3$, and $q^*_{j\bar{k}\bar{t}_3} = f^*_{a_{j\bar{k}\bar{t}_3}} \geq x_{p_{i\bar{t}_1jtj'\bar{k}\bar{t}_3}} > 0$. From the second summand of (62) it then follows that since $q^*_{\bar{k}\bar{t}_3} > 0$, $\gamma_{2j\bar{k}\bar{t}_3} = \rho_{2j\bar{t}_3} = \rho^*_{3\bar{k}\bar{t}_3} - \hat{c}_{j\bar{k}\bar{t}_3}(Q^{2*}_{\bar{t}_3})$. From the third summand of (62), we know that since $u_{j\tau} \geq x_{p_{i\bar{t}_1jtj'\bar{k}\bar{t}_3}} > 0, t \leq \tau \leq \bar{t}_3$, we have $\gamma_{2jt} = \gamma_{j\bar{k}\bar{t}_3} - \sum_{\tau=t}^{\bar{t}_3-1} \frac{\partial cv_{j\tau}(u^*_{j\tau})}{\partial u_{j\tau}}$. Therefore, $\rho^*_{1ijt} = \rho^*_{3\bar{k}\bar{t}_3} - \hat{c}_{j\bar{k}\bar{t}_3}(Q^{2*}_{\bar{t}_3}) - \sum_{\tau=t}^{\bar{t}_3-1} \frac{\partial cv_{j\tau}(u^*_{j\tau})}{\partial u_{j\tau}} - \frac{\partial c_{jt}(Q^{1*})}{\partial q_{ijt}}$.

Note that the value of ρ^*_{1ijt} can be recovered from any active path that contains link a_{ijt}. Indeed, such a pricing mechanism is crucial and guarantees that the sum of (5), (11), and (15), which yields variational inequality (16), corresponds to an equilibrium in the sense of Definition 1.